Dense electron beam radiation in a dielectric medium

S. G. Oganesyan
R&D Company "Lazerayin Tekhnika," Yerevan 375 090, Armenia
(Received 3 July 1996; revised manuscript received 27 May 1997)

We have studied both the spectral intensity and the energy of radiation produced by an electron beam in dielectric medium. Since the device starts up from a photon vacuum the investigation is based on a quantum electrodynamics approach. We have assumed that the electron beam longitudinal size is indefinite and its energy spread is of Gaussian form. In such a case the radiation process has a cutoff with respect to the electron density ρ_0. If the electron density is lower than some characteristic density $\rho_{\rm ch}$ then the radiation will be mainly spontaneous and its emission line will have Gaussian shape. If the electron beam is dense, $\rho_0 > \rho_{\rm ch}$, then a drastic change of line form takes place. Namely, one observes a high and narrow peak (or superradiation effect) on its right-hand side and radiation energy saturation on its left-hand side. Both of these phenomena are due to the stimulated processes of photon absorption and emission. We have evaluated the level of radiation fluctuations in the peak vicinity. Numerical estimations of both obtained effects are presented. [S1063-651X(97)07709-X]

PACS number(s): 41.60.Bq

I. INTRODUCTION

The theory of the Vavilov-Cherenkov effect based on Maxwell equations was developed by Tamm and Frank [1]. As a matter of fact, they calculated the radiation produced by an electron traversing a dielectric medium. They showed that spectral distribution of Cherenkov radiation has δ-type form [i.e., spectral intensity $dI \sim \delta(\omega - kv\cos\theta)\, do\, d\omega$], so that the frequency ω emitted at the θ angle is defined by

$$1 - n\beta\ \cos\theta = 0. \tag{1}$$

Here $n(\omega)$ is the refractive index of the medium, $\beta = v/c$, v is the electron velocity, c is the light velocity, θ is the angle between wave vector $\mathbf{k}$ and velocity $\mathbf{v}$. The calculations based on quantum electrodynamics (QED) lead to a similar result [of course, not for purely quantum effects, such as recoil or electron spin moment contribution, see Eq. (34)]. Nevertheless, it should be emphasized that the classical description of an electromagnetic field holds true only if its electric field strength $\mathbf{E}$ is high enough [2],

$$|\mathbf{E}| \gg \frac{\omega^2}{c^2}\ \sqrt{\hbar c}, \tag{2}$$

or if the photon number per unit volume is large enough, $N/V \gg 1$.

In the present paper we consider radiation produced by an electron beam (e-beam). If the electron density is high enough, then such a device might be used as a cm-mm wavelength source of electromagnetic radiation. On the basis of our result one may estimate the role of spontaneous noise in a Cherenkov amplifier and develop the theory of a Cherenkov oscillator. It should also be noted, that e-beam radiation can be used for its own structure investigation. To simplify the problem one can differentiate two cases.

(1) e-bunch size l is of order or less than radiation wavelength $\lambda (l \leqslant \lambda)$. In such a case all bunch particles radiate as one particle. As a result, the radiated field quantities are directly proportional to the number of electrons, $N^{(e)}$, whereas the intensity of the radiation is directly proportional to $(N^{(e)})^2$. This effect in the cases of Cherenkov radiation [3], transition radiation [4], and undulator radiation [5] was recently observed in mm wavelengths region. Note, that it is a very complicated problem to generate a very short electron bunch.

(2) In the present paper we consider the opposite case $l \gg \lambda$ (or where the e-beam longitudinal size is indefinite). The major feature of such a device is that a photon, produced by an electron, moves in e-beam for a long time. As a result, the possibilities of stimulated emission as well as absorption appear. We show that both these phenomena lead to drastic changes of spontaneous radiation line form. It will be noted here that the pure stimulated effects (in fact, stimulated Cherenkov effects) were studied in many of our other papers and generalized in review [6], Chaps. I and II. In our works, based on Maxwell-Dirac equations for c-number functions, we always assumed that our device (e-beam–dielectric medium) contains a probing signal (classical monochromatic wave).

In the present paper we consider the case where the same device contains no probing wave (or starts up from a photon vacuum). Therefore, the radiation field is built up from spontaneous noise. For simplicity we suppose that the e-beam only has an energy spread of Gaussian form. If the e-beam density is small then one can neglect the stimulated processes contribution. In such a case the radiation line form also has Gaussian form. But when the e-beam density exceeds some critical value, then the drastic changes of line form take place. Namely, on its right-hand side the process of stimulated emission forms a high narrow peak—an effect of Cherenkov superradiation. The next remarkable phenomenon is possible on the left-hand side of radiation line. Here (within the frequency finite region) the radiation and the absorption processes may compensate each other. We have associated this photon equilibrium state with the well-known black radiation and called it Cherenkov black radiation. It should be emphasized that the latter effect is possible only if one takes spontaneous emission into account.

The organization of the remainder of this paper is as follows. In Sec. II, we write the Hamiltonian of our system. In Sec. III, we calculate the energy and the intensity of radiation produced by the e-beam using the Schrödinger picture. In Sec IV, we obtain the same quantities more exactly on the basis of the Heisenberg picture. In Sec. V, the peculiarities of radiation line shape are studied. In Sec. VI, the question of radiation fluctuation level is discussed. And, finally, in Sec. VII, we consider the possibility of experimental observation of superradiation effects.

II. THE SYSTEM HAMILTONIAN

Consider now the interaction of the electron field with the electromagnetic field in the dielectric medium. To simplify the problem we assume, as usual [2], that the coupling are in a large cube of $V=L^3$ volume and impose periodic boundary conditions. Then the components of the electromagnetic field wave vector are

$$k_x=\frac{2\pi}{L}n_x,\quad k_y=\frac{2\pi}{L}n_y,\quad k_z=\frac{2\pi}{L}n_z, \tag{3}$$

where $n_{x,y,z}=0,\pm1,\pm2\ldots$. For simplicity we suppose that dielectric medium permeability $\mu=1$, whereas permitivity $\varepsilon(\omega)$ is a slowly varying function of ω frequency. Taking into account that the electromagnetic field total energy is

$$W=\frac{1}{8\pi}\int(\varepsilon\mathbf{E}^2+\mathbf{H}^2)dV,$$

we may carry out the well-known second quantization procedure [2]. Then the field vector potential operator is

$$\mathbf{A}=\sum_{\mathbf{k}}\sum_{\alpha=1}^{2}\sqrt{\frac{2\pi\hbar c^2}{V\omega\varepsilon}}[c_{\mathbf{k}\alpha}\mathbf{e}^{(\alpha)}e^{i\mathbf{k}\cdot\mathbf{r}}+c_{\mathbf{k}\alpha}^{\dagger}\mathbf{e}^{(\alpha)}e^{-i\mathbf{k}\cdot\mathbf{r}}]. \tag{4}$$

Here ε is the average value of the dielectric constant, $c_{\mathbf{k}\alpha}$ and $c_{\mathbf{k}\alpha}^{\dagger}$ are photon annihilation and creation operators in state with the $\mathbf{k}$ wave vector, and $\mathbf{e}^{(\alpha)}$ is a unit polarization vector.

Consider the second quantization of the electron field. We neglect the influence of electron multiple scattering by the molecular of medium and assume that the radiated field strength is not too high. Then the ψ operator expression is [2]

$$\psi=\sum_{\mathbf{p}}\sum_{\sigma=-1/2}^{1/2}\frac{1}{\sqrt{2VE}}[a_{\mathbf{p}\sigma}u_{p\sigma}e^{(i/\hbar)\mathbf{p}\cdot\mathbf{r}}+b_{\mathbf{p}\sigma}^{\dagger}u_{-p-\sigma}e^{-(i/\hbar)\mathbf{p}\cdot\mathbf{r}}]. \tag{5}$$

Here $a_{\mathbf{p}\sigma}$ and $b_{\mathbf{p}\sigma}^{\dagger}$ are the electron annihilation and the positron creation operators in state with $\mathbf{p}$ momentum and σ helicity, four-component spinors $u_{p\sigma}$ and $u_{-p-\sigma}$ are normalized so that $\overline{u}_{\pm p\sigma}u_{\pm p\sigma}=\pm2mc^2$ (here $\overline{u}=u^{\dagger}\gamma^0$ is an adjoint bispinor, m is the electron mass, and γ^0 is the Dirac matrix).

Furthermore, we shall study only the photon expectation occupation number (ON) in lth state, namely, when $k_x=k_y=0$, $k=k_z=k_l=2\pi l/L$, $\omega=\omega_l=ck_l/n$, and $a=1$, i.e., when the photons are polarized along x axis. These photons interact with electrons whose momentum vector is placed in the xz coordinate plane and is parallel to the z' axis. Since the volume V is arbitrary large, then, due to the periodic boundary conditions, the electron momentum components are

$$p_{x'}=p_y=0,\quad p_{z'}=p_j=\frac{2\pi\hbar}{L}j,$$

where

$$j=0,1,2\ldots\ . \tag{6}$$

According to general principles, our system evolution is described by the Schrödinger equation

$$i\hbar\frac{\partial|\phi\rangle}{\partial t}=(H_e+H_{\rm ph}+H_{\rm int})|\phi\rangle, \tag{7}$$

where

$$H_e=\sum_r E_r(a_r^+a_r+b_r^+b_r),\quad H_{\rm ph}=\sum_i\hbar\omega_ic_i^{\dagger}c_i \tag{8}$$

are the Hamiltonians of the free electrons and positrons and free photons, whereas

$$H_{\rm int}=\frac{1}{c}\int j_{\mu}A^{\mu}dV \tag{9}$$

is the Hamiltonian of their interaction and

$$j^{\mu}=ec\hat{N}(\overline{\psi}\gamma^{\mu}\psi) \tag{10}$$

is the operator of current density. For simplicity, we suppose further that photon expectation ON $N\ll2mc^2/\hbar\omega$. In such a case one may omit positron operators in Eqs. (5) and (8) and normal product operator $\hat{N}$ in Eq. (10).

III. SCHRÖDINGER PICTURE

In this section we consider photon expectation ON in the Schrödinger picture. The latter supposes that one may obtain the system wave function and, as a result, the probability amplitudes $\varphi_N(t)$ (as well as probabilities $W_N(t)=|\varphi_N|^2$) for finding N photons in the system. Then the photon expectation ON at a given time t is

$$\overline{N}=\sum_{N=0}^{\infty}NW_N(t). \tag{11}$$

Nevertheless, the computation of the exact system wave function seems to be a complicated enough problem. The simplest (but rough) way to avoid this difficulty is connected with the balance equation. We shall derive it in Sec. III A. In Sec. III B we shall calculate the e-beam spontaneous emission. The modification of this radiation due to the stimulated processes will be considered in Sec. III C.

A. Balance equation

First of all, we assume that the system already contains N photons and $|N\rangle$ is their wave function. We denote by $|1_1,0_2\rangle,|0_1,1_2\rangle$ and by $|1_3,0_4\rangle,|0_3,1_4\rangle$ the initial and the final wave functions of the electrons which take part in radiation and absorption processes

$$p_1^\mu - \hbar k^\mu = p_2^\mu, \quad p_3^\mu + \hbar k^\mu = p_4^\mu. \tag{12}$$

Here p^μ and k^μ are the electron momenta (contrvariant) fourth vector and photon fourth wave vector, $\mu=0,1,2,3,4$. It is clear, that the simplest wave function, which will conjuncture both these reactions, contains two electrons, at least. Therefore, we write

$$|\phi_0\rangle = |N,1_1,0_2,1_3,0_4\rangle. \tag{13}$$

Note, that in such a description the electron ON-*s* in 1-*t* and 3-*d* states $N_1^{(e)}$ and $N_3^{(e)}$, are maximal and equal to 1 according to Fermi's principle (below we shall take into account that e-beam expectation ON-*s* $N_1^{(e)} \neq N_3^{(e)}$, and that their values are less than unit). It is interesting to note that this is not the case at the classical level. Here the number of electrons in any state may be arbitrary large and the possibility of unsuccessful results appears.

Now we confine ourselves to linear field approximations. We substitute $|\Phi_0\rangle$ for $|\Phi\rangle$ to the right of Eq. (7) and integrate over time within the $[-T/2,T/2]$ interval. If $T=\infty$ then the system final wave function is

$$|\phi\rangle = \varphi_N|N,1_1,0_2,1_3,0_4\rangle + \varphi_{N+1}|N+1,0_1,1_2,1_3,0_4\rangle + \varphi_{N-1}|N-1,1_1,0_2,0_3,1_4\rangle. \tag{14}$$

Here

$$\varphi_{N+1} = \frac{(2\pi)^4 e\hbar^3 c}{iV}\left[\frac{(N+1)\pi\hbar}{2E_1E_2V\omega\varepsilon}\right]^{1/2} \times \overline{u_2}\hat{e}^{(1)}u_1\delta^{(4)}(p_1-\hbar k-p_2),$$

$$\varphi_{N-1} = \frac{(2\pi)^4 e\hbar^3 c}{iV}\left[\frac{N\pi\hbar}{2E_3E_4V\omega\varepsilon}\right]^{1/2} \times \overline{u_4}\hat{e}^{(1)}u_3\delta^{(4)}(p_3+\hbar k-p_4), \tag{15}$$

whereas the φ_N amplitude is obtained from the condition of probability conservation

$$W_N + W_{N+1} + W_{N-1} = 1 \tag{16}$$

(here $\hat{e}^{(1)} = e_\mu^{(1)}\gamma^\mu = -\gamma^1$). During the calculations we have taken into account that the interaction Hamiltonian terms (9), which lead to reactions (12), are of two types only;

$$\delta^{(4)}(p_i - \hbar k - p_f)a_f^\dagger a_i c^\dagger|\phi_0\rangle$$

and

$$\delta^{(4)}(p_i + \hbar k - p_f)a_f^\dagger a_i c|\phi_0\rangle.$$

The first type deals with $i=1$ and $f=2$ indices, whereas the second one deals with $i=3$ and $f=4$ indices. The contributions of all other terms vanish because of conservation laws (12).

Now we introduce the expectation probabilities of photon radiation and absorption per unit time

$$w_{N\pm1} = \frac{W_{N\pm1}}{T}. \tag{17}$$

Here T is the time of interaction. Note, that one might substitute the $VT/(2\pi)^4$ expression for one of the $\delta^{(4)}$ functions and perform exact division. Now, according to the meaning of the w_N value, we write a balance equation for photon number change dN during the time dt,

$$dN = w_{N+1}dt - w_{N-1}dt$$

or

$$\frac{dN}{dt} = w_{N+1} - w_{N-1}. \tag{18}$$

Further (Sec. 6) we shall also need an equation for the second moment of the N value $\overline{N^2}$. One can derive it in the following formal way. As it follows from Eq. (17) and Eq. (16) the probabilities $W_{N+1}(t)$ and $W_N(t)$ at the time t are (approximately) $W_{N\pm1}(t) = (t+T/2)w_{N\pm1}$, $W_N(t) = 1 - W_{N+1}(t) - W_{N-1}(t)$ [note that we denoted $W_{N\pm1}(t=T/2)$ by $W_{N\pm1}$]. Then the N^2 expectation value $\overline{N^2}$ is

$$\overline{N^2} = \sum_{n=N-1}^{N+1} n^2 W_n(t).$$

Now we carry out time differentiation of both sides of this equation and substitute $\overline{n^2}$ and $\overline{n}$ for n^2 and n in the right-hand side. Finally, we have

$$\frac{d\overline{N^2}}{dt} = \overline{(N-1)^2}w_{\overline{N}-1} + \overline{N^2}w_{\overline{N}} + \overline{(N-1)^2}w_{\overline{N}+1}. \tag{19}$$

The same calculations for the first power of the N variable leads to Eq. (18).

Before examination of Eqs. (18) and (19) consider the transition procedure to the real e-beam. Note, first, that the obtained probabilities

$$w_{N\pm1}(p_i,p_f) \sim \delta^{(4)}(p_i \mp \hbar k - p_f)Tr[(u\overline{u})_i(u\overline{u})_f], \tag{20}$$

where $i=1$ and $f=2$ for the $N+1$ case, whereas $i=3$ and $f=4$ for the $N-1$ case; spinor indices are omitted for simplicity. If the e-beam is monoenergetic but arbitrary polarized, then one has to introduce a density matrix. According to [2] we must substitute

$$\frac{1}{2}(\hat{p}c + mc^2)(1-\gamma^5\hat{a}) \quad \text{for } u\overline{u} \tag{21}$$

in Eq. (20). Here $\hat{p} = p^\mu\gamma_\mu$, $\hat{a} = a^\mu\gamma_\mu$, a^μ is fourth polarization vector.

If we are not interested in the e-beam final state, then we are to sum over the electron all possible states. Since the e-beam density matrix (21) and the interaction takes place in large V volume, then such a procedure is equivalent to substitution

$$(\hat{p}_f c + mc^2)\,\frac{Vd\mathbf{p}_f}{(2\pi\hbar)^3} \tag{22}$$

for $(u\bar{u})_f$ in Eq. (20). The second step is to introduce the parameters of the initial e-beam, such as density ρ_0, polarization a^μ, and energy spread Δ. For simplicity, we restrict ourselves to the case of the unpolarized e-beam. Then $a^\mu = 0$ and we must substitute

$$\frac{1}{2}\,(\hat{p}_i c + mc^2) \tag{23}$$

for $(u\bar{u})_i$ in Eq. (20). Assume that there are $N^{(e)}$ electrons in V volume and that $N_j^{(e)}$ is electron expectation ON in the jth state, so that $N_j^{(e)} \leqslant 1$, $\Sigma N_j^{(e)} = N^{(e)}$, and $\rho_0 = N^{(e)}/V$ is the e-beam density. According to general principles, the total probabilities of photon radiation or absorption per unit time are

$$w_{N\pm 1}^{(t)} = \sum_i N_i^{(e)} w_{N\pm 1}(p_i). \tag{24}$$

As was mentioned earlier interaction volume V is large. Therefore, the allowed values of electron momentum p_i (6) [as well as energy values $E_i = (p_i^2 c^2 + m^2 c^4)^{1/2}$] are very close to one another. In other words, the e-beam momentum and energy spectra, in fact, are continuous functions of the $\mathbf{p}$ and E variables [note, that the same is also true for the e-beam final state (22)]. We assume, for definitness, that both distribution functions $N^e(E)$ *and* $N^{(e)}(\mathbf{p})$ have Gaussian form; namely,

$$N^{(e)}(E) = \rho_0 V f(E),$$

$$N^{(e)}(\mathbf{p}) = \rho_0 V \delta(p_{x'})\,\delta(p_y) N^{(e)}(p_{z'}), \tag{25}$$

where

$$f(E) = \sqrt{\frac{4\ln 2}{\pi}}\,\frac{1}{\Delta}\,\exp\left[-4\ln 2\,\frac{(E-E_0)^2}{\Delta^2}\right],$$

$$N^{(e)}(p_{z'}) = N^{(e)}(E)\,\frac{dE}{dp_{z'}}. \tag{26}$$

Here E_0 is the average energy of the e-beam and Δ is the width of the energy spread. Recall, that in the present paper we are not interested in the e-beam angular spread (some spin and angular spread effects were discussed in our review [6], Secs. 12, 10, and 14). Now the well-known transition procedure from summation to integration can be carried out:

$$\sum_j N_j^{(e)} w_{N\pm 1}(\mathbf{p}_j) \approx \sum_j N^{(e)}(\mathbf{p}_j)\,\frac{Vd\mathbf{p}_j}{(2\pi\hbar)^3}\,w_{N\pm 1}(\mathbf{p}_j) \rightarrow \int N^{(e)}(\mathbf{p}_i) w_{N\pm 1}(\mathbf{p}_i)\,\frac{Vd\mathbf{p}_i}{(2_N\hbar)^3}. \tag{27}$$

Here $i=1$ for the $N+1$ case and $i=3$ for the $N=1$ one. Taking into account Eqs. (22), (23), and (27) we can represent the $w_{N\pm 1}$ probabilities as

$$w_{N+1} = (N+1) R_1 f(E_1), \quad w_{N-1} = N R_3 f(E_3) \tag{28}$$

and rewrite the balance equation in a clearer form:

$$\frac{dN}{dt} = N[R_1 f(E_1) - R_3 f(E_3)] + R_1 f(E_1). \tag{29}$$

Note that (as usual) the w_{N+1} expression contains both stimulated (term $\sim N$) and spontaneous (term ~ 1) emissions, whereas the w_{N-1} expression contains only the stimulated process. The first term in the right-hand side of the Eq. (29) describes the competition of stimulated emission and stimulated absorption processes, whereas the second one is responsible for spontaneous emission. The multiplicative factors $R_{1,3}$ are defined as

$$R_{1,3} = \frac{2\pi c\lambda\rho_0 e^2 \beta_{1,3}^2 E_{1,3}\sin^2\theta)}{\hbar\omega\varepsilon}\left(\frac{p_{1,3}}{mc}\right)^2 \times\left[1 \mp \frac{\hbar\omega(1-n\beta_{1,3}\cos\theta)}{2E_{1,3}\beta_{1,3}^2\sin^2\theta}\right]. \tag{30}$$

The values

$$E_{1,3} = E_c \pm \Delta E \tag{31}$$

are those electrons energies which take part in radiation and absorption processes and

$$E_c = \frac{mc^2}{[1-(n\ \cos\theta)^{-2}]^{1/2}}$$

and

$$\Delta E = \frac{1}{2}\,\hbar\omega(n^2-1)\left(\frac{p_c}{mc}\right)^2 \tag{32}$$

are the classical and the quantum parts of these energies obtained from Eqs. (12).

B. *e*-beam spontaneous radiation

In this limit we neglect the first term in the right-hand side of Eq. (29). Suppose that our system starts up from photon vacuum at the time $t=t_0$, i.e., $N(t_0)=0$. Then at the time t photon expectation ON is

$$N = (t-t_0) R_1 f(E_1).$$

Now one may introduce radiated field spectral energy $dW=\hbar\omega N\ Vd\mathbf{k}/(2\pi)^3$ and intensity $dI=dW/dt$. Taking into account that $d\mathbf{k}=(n/c)^3\omega^2 d\omega d0$ ($d0=\sin\vartheta d\theta d\varphi$ is a solid angle) and Eq. (30) we have

$$dI=N^{(e)}\frac{n}{2\pi c}e^2\omega\beta_1^2\sin^2\theta E_1\left(\frac{p_1}{mc}\right)^2 f(E_1)\times\left[1-\frac{\hbar\omega(1-n\beta_1\cos\theta)}{2E_1\beta_1^2\sin^2\theta}\right]dOd\omega,$$

$$dW=(t-t_0)dI. \tag{33}$$

Here $\beta_1=v_1/c$, $p_1=E_1v_1/c^2$, and E_1 energy is defined by Eqs. (31) and (32).

It seems useful to derive the above expressions as a direct generalization of the Tamm-Frank formula. In the case of one particle the radiation spectral intensity is $dI=\hbar\omega w_1Vd\mathbf{k}/(2\pi)^3$. Here the w_1 value is defined by w_{N+1} expression (28) at $N=0$. Note that the $|1_3\rangle$ state does not play any role in the production of ω frequency (12). If we are not interested in the electron final state and the electron initial state is not polarized, then the substations (22) and (23) should be carried out. Finally, we have

$$dI=\frac{n}{2\pi c}e^2\omega^2\beta^2\sin^2\theta\left[1-\frac{\hbar\omega(1-n\beta\cos\theta)}{2E\beta^2\sin^2\theta}\right]\times\delta\left(\omega-kv\cos\theta+\frac{\hbar\omega(n^2-1)}{2E}\right)do\omega,$$

$$dW=(t-t_0)dI. \tag{34}$$

Note, that the multiplicative factor in square brackets does not appear during calculations based on the Klein-Gordon equation. Hence it is due to the electron spin moment. When we neglect the latter effect and the recoil term in the argument of the δ function as well, then Eqs. (34) coincide with the same ones derived by Tamm and Frank. The averaging over the e-beam energy spread $N^{(e)}f(E)$ leads to Eq. (33). Thus the e-beam spontaneous radiation is directly proportional to interaction volume V and electron density ρ_0. Note also that now, in contrast to the one particle case (34), the radiation intensity and energy are smoothly varying functions of ω frequency.

C. Dense electron beam radiation

Now we can consider the modification of spontaneous radiation intensity and energy due to the stimulated processes. First of all, we simplify the right-hand side of Eq. (29). We neglect the electron spin moment contribution and take into account that $R_{1,3}$ are slowly varying functions of energy, so that

$$R_{1,3}\approx\frac{2\pi c\lambda\rho_0e^2\beta_0^2E_0\sin^2\theta}{\hbar\omega\varepsilon}\left(\frac{p_0}{mc}\right)^2.$$

Since the quantum correction ΔE (32) is small ($\Delta E\ll E$), then $f(E_1)-f(E_3)=2(\partial f/\partial E_c)\Delta E$. Finally, we have

$$\frac{dN}{d\tau}=GN+Q. \tag{35}$$

It should be noted that replacing the t variable by a new one, $\tau=ct/n$, makes it possible in our problem to pass from the time picture to the space one. The value

$$G=-32\sqrt{\pi}(\ln 2)^{3/2}\rho_0r_0\lambda\beta_0^3\sin^2\theta\,\frac{n^2-1}{n}\times\left(\frac{p_0}{mc}\right)^3\left(\frac{E_0}{\Delta}\right)^2\frac{E_c-E_0}{\Delta}\exp\left[-4\ln 2\frac{(E_c-E_0)^2}{\Delta^2}\right] \tag{36}$$

is completely equal to the Cherenkov laser gain obtained by us earlier ([6], Sec. 10) (here $r_0=e^2/mc^2$ is the classical electron radius). The value

$$Q=4\sqrt{\pi\ln 2}\rho_0r_0\lambda\beta_0^2\sin^2\theta\,\frac{1}{n}\left(\frac{p_0}{mc}\right)^2\frac{E_0}{\Delta}\frac{mc^2}{\hbar\omega}\times\exp\left[-4\ln 2\frac{(E_c-E_0)^2}{\Delta^2}\right] \tag{37}$$

describes the source of spontaneous radiation. Note that the $G(E_c)$ function achieves its maximum $G_{\max}=G_1$ and minimum $G_{\min}=-G_2$ values at

$$E_c=E'_{1,2}=E_0\mp\frac{\Delta}{\sqrt{8\ln 2}} \tag{38}$$

energies. The $G_{1,2}$ values are defined as

$$G_{1,2}=8{,}4\rho_0r_0\lambda_{1,2}\beta_0^3\sin^2\theta\,\frac{n_{1,2}^2-1}{n_{1,2}}\left(\frac{p_0}{mc}\right)^3\left(\frac{E_0}{\Delta}\right)^2, \tag{39}$$

where $\omega_{1,2}$ frequencies (and $\lambda_{1,2}=2\pi c/\omega_{1,2}$ wavelengths) are determined from the

$$1-n(\omega)\beta'_{1,2}\cos\theta=0$$

equations, $\beta'_{1,2}=cp'_{1,2}/E'_{1,2}$. Omitting the GN term in Eq. (35) we return to the previous case. If we neglect the Q term, then Eq. (35) coincides with that for the Cherenkov laser [6]. Note, that the latter has a nontrivial solution only if the device contains a probing wave from the outset.

Thus, in contrast to the consideration based on Maxwell-Dirac equations for c-number functions [6], the one based on QED allows us to take into account spontaneous emission. If our system starts up from photon vacuum at the time $t=0$, then at the time t

$$N=\frac{Q}{G}[\exp(G\tau)-1]. \tag{40}$$

Therefore, the energy and the intensity of radiation produced by the e-beam are, respectively,

$$dW = \hbar\omega \frac{Q}{G}[\exp(G\tau)-1]\frac{Vn^3\omega^2 d\omega do}{(2\pi c)^3},$$

$$dI = \frac{\hbar\omega c}{n} Q \exp(G\tau)\frac{Vn^3\omega^2 d\omega do}{(2\pi c)^3}. \tag{41}$$

Note, that the Q/G fraction is an universal value determined only by conservation laws (12) and distribution function (26)

$$\frac{Q}{G} = \frac{f(E_1)}{f(E_1)-f(E_3)} \approx -\frac{1}{8\ln 2}\frac{\Delta}{E_c-E_0}\frac{\Delta}{\hbar\omega}\left(\frac{mc}{p_0}\right)^2\frac{1}{n^2-1}. \tag{42}$$

When the $G\tau$ parameter is small ($G\tau \ll 1$) then the radiation has a mainly spontaneous nature.

If $G\tau \gg 1$, then the e-beam emission may increase sharply due to the exponential term (see Sec. V). It is interesting to note that the radiation energy and intensity (41) are directly proportional to coupling volume V (as in the spontaneous radiation case), but exponentially depend on e-beam density ρ_0. Such dependence is in contrast both to the spontaneous radiation case (33) and the coherent radiation case [3–5]. If our device operates in the amplifier regime, then the solution of Eq. (35) is

$$N = N_0 \exp(G\tau) + \frac{Q}{G}[\exp(G\tau)-1].$$

Here N_0 is the number of amplified wave photons at the $t=0$ time. It is clear, that the first term describes an exponential growth of the initial signal, whereas the second one describes the spontaneous noise of amplifier. It is obvious, that the

$$N_0 \gg \frac{Q}{G} \tag{43}$$

inequality is the necessary condition of amplifier operation [the Q/G fraction is determined by Eq. (42)].

IV. HEISENBERG PICTURE

It seems, that the most suitable tool for studying stimulated processes is the Heisenberg picture (HP), since the system of equations for field operators is completely similar to that for c-number functions. In this section we denote by $\hat{L}$ and L the Heisenberg and the Schrödinger operators, respectively. Note that in the HP the photon expectation ON is

$$N = \langle \phi_0 | \hat{c}^\dagger_{\mathbf{k}\alpha}\hat{c}_{\mathbf{k}\alpha} | \phi_0 \rangle, \tag{44}$$

where

$$|\phi_0\rangle = |0_\gamma, 1_1, 0_2, 1_3, 0_4\rangle \tag{45}$$

is our device wave function at the time $t=t_0$, $\hat{c}^\dagger_{\mathbf{k}\alpha}$ and $\hat{c}_{\mathbf{k}\alpha}$ are electromagnetic field operators in the HP. Now we derive the system of equations for the photon $\hat{c}$ and $\hat{c}^\dagger$ operators and for the electron $\hat{a}_r$ and $\hat{a}^\dagger_r$ ones (here we omitted photon operator indices; the r index stands for $\mathbf{p}\sigma$ indices in the electron operators). Using the equation of motion for Heisenberg operator $\hat{L}$

$$\frac{d\hat{L}}{dt} = \frac{i}{\hbar}[\hat{H}\hat{L}]$$

the Hamiltonian definition (8) and (9) and commutative rules for Schrödenger operators

$$cc^\dagger - c^\dagger c = 1, \quad a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij},$$

we obtain

$$\frac{d\hat{c}}{dt} = -i\omega\hat{c} - \sum_{i,j} \hat{a}_i^\dagger \hat{a}_j I_{ij}^-,$$

$$\frac{d\hat{c}^\dagger}{dt} = i\omega\hat{c}^\dagger + \sum_{i,j} \hat{a}_i^+ \hat{a}_j I_{ij}^+, \tag{46}$$

$$\frac{d\hat{a}_r}{dt} = -\frac{i}{\hbar}E_r\hat{a}_r - \sum_f (\hat{a}_f\hat{c}I_{rf}^\dagger + \hat{a}_f\hat{c}^\dagger I_{rrf}^-),$$

$$\frac{d\hat{a}_r^\dagger}{dt} = \frac{i}{\hbar}E_r\hat{a}_r^\dagger + \sum_f (\hat{a}_f^\dagger\hat{c}I_{fr}^\dagger + \hat{a}_f^\dagger\hat{c}^\dagger I_{fr}^-). \tag{47}$$

Here the coupling constants $J_{ij}^{\pm}$ are defined as

$$I_{ij}^{\pm} = \frac{i(2\pi)^3 e\hbar^2 c}{V}\left(\frac{\pi\hbar}{2E_iE_jV\omega\varepsilon}\right)^{1/2}\overline{u_i\hat{e}}^{(1)}u_j\delta(\mathbf{p}_j \pm \hbar\mathbf{k} - \mathbf{p}_i). \tag{48}$$

Note that at the time t_0

$$\hat{c}(t_0) = c, \quad \hat{c}^\dagger(t_0) = c^\dagger, \quad \hat{a}_r(t_0) = a_r, \quad \hat{a}_r^\dagger(t_0) = a_r^\dagger, \tag{49}$$

and $N(t_0)=0$. As it follows from the set (46) it is convenient to introduce new operators $\mathbf{c}, \mathbf{c}^\dagger$, $\mathbf{a}, \mathbf{a}^\dagger$ so that

$$\hat{c} = \mathbf{c}\exp(-i\omega t), \quad \hat{c}^\dagger = \mathbf{c}^\dagger\exp(i\omega t),$$

$$\hat{a}_r = \mathbf{a}_r\exp(-iE_rt/\hbar), \quad \hat{a}_r^\dagger = \mathbf{a}_r^\dagger\exp(iE_rt/\hbar). \tag{50}$$

Then the basic system of equations, which describes our device is

$$\frac{d\mathbf{c}}{dt} = -\sum_{i,j} \mathbf{a}_i^\dagger\mathbf{a}_j I_{ij}^- \exp(-i\Delta E_{ij}^- t/\hbar),$$

$$\frac{d\mathbf{c}^\dagger}{dt} = \sum_{i,j} \mathbf{a}_i^\dagger\mathbf{a}_j I_{ij}^+ \exp(-i\Delta E_{ij}^+ t/\hbar),$$

$$\frac{d\mathbf{a}_r}{dt} = -\sum_f [\mathbf{a}_f\mathbf{c}I_{rf}^+\exp(-i\Delta E_{rf}^+ t/\hbar) + \mathbf{a}_f\mathbf{c}^\dagger I_{rf}^- \times\exp(-i\Delta E_{rf}^- t/\hbar)], \tag{51}$$

$$\frac{d\mathbf{a}_r^\dagger}{dt} = \sum_f [\mathbf{a}_f^\dagger\mathbf{c}I_{fr}^+\exp(-i\Delta E_{fr}^+ t/\hbar) + \mathbf{a}_f^\dagger\mathbf{c}^\dagger I_{fr}^- \times\exp(-i\Delta E_{fr}^- t/\hbar)], \tag{52}$$

where the energy detuning

$$\Delta E_{if}^{\mp} = E_f \mp \hbar\omega - E_i. \quad (53)$$

Note that there now appears the possibility to assume that in some cases new field operators may be slowly varying functions of time with respect to the exponents. We shall study the solutions of the obtained system in two cases: spontaneous radiation and dense e-beam radiation.

A. Spontaneous radiation

In this limit we neglect photon operators ''influence'' on electron operators in Eqs. (52) by setting the right-hand side of this equation equal to zero. Accounting Eq. (49) gives

$$\mathbf{a}_r = a_r, \quad \mathbf{a}_r^\dagger = a_r^\dagger. \quad (54)$$

To simplify the calculations we suppose that the time $t_0 = -\infty$. Then we substitute a and $a^\dagger$ operators for $\mathbf{a}$ and $\mathbf{a}^\dagger$ operators in the right-hand side of Eq. (51) and perform time integration within the $[-\infty, +\infty]$ interval. Finally, we express new photon operators in terms of photon and electron Schrödinger operators:

$$\mathbf{c} = c - \sum_{i,j} a_i^\dagger a_j I_{ij}^- R_{ij}^-, \quad \mathbf{c}^\dagger = c^\dagger + \sum_{r,f} a_r^\dagger a_f I_{rf}^+ R_{rf}^+. \quad (55)$$

Here the multiplicative factors $R_{ij}^{\pm}$ are

$$R_{ij}^{\pm} = 2\pi\hbar\,\delta(E_j \pm \hbar\omega - E_i). \quad (56)$$

Now we may introduce Eq. (55) into Eq. (44) and pass to the real e-beam using Eqs. (22), (23), and (27). Finally, as one might easily predict, we come to Eqs. (33).

B. Dense *e*-beam radiation

The above calculations have shown that electron operators approximation by Schrödinger operators in Eq. (51) leads only to the account of spontaneous emission. Now we solve Eqs. (52) more exactly. To simplify the problem, we approximate the $\mathbf{a}, \mathbf{a}^\dagger$ operators using the $a, a^\dagger$ ones in the right-hand side of Eq. (52) and suppose that the $\mathbf{c}$ and $\mathbf{c}^\dagger$ operators are slowly varying functions of time [note, that in Cherenkov laser theory (Ref. [6], Sec. 10) a similar assumption corresponds to the small gain approximation which holds true when the e-beam density is not too high]. Then one obtains

$$\mathbf{a}_r = a_r + \Delta_1, \quad \mathbf{a}_r^\dagger = a_r^\dagger + \Delta_2, \quad (57)$$

where the $\Delta_{1,2}$ operators are, respectively,

$$\Delta_1 = -\sum_f a_f(\mathbf{c} I_{rf}^+ R_{rf}^+ + \mathbf{c}^\dagger I_{rf}^- R_{rf}^-),$$

$$\Delta_2 = \sum_f a_f^\dagger(\mathbf{c} I_{fr}^+ R_{fr}^+ + \mathbf{c}^\dagger I_{fr}^- R_{fr}^-). \quad (58)$$

The multiplicative factors

$$R_{ij}^{\pm} = \int_{t_0}^{t} dt\, \exp\left(-\frac{i}{\hbar}\Delta E_{ij}^{\pm} t + \eta t\right) \quad (59)$$

and the vanishing value η switches off the electron-photon coupling at the time $t_0 = -\infty$. Now one may introduce Eq. (58) into Eq. (51) and obtain the system of equations for photon operators:

$$\frac{d\mathbf{c}}{dt} = \mathbf{c} g_1' + \nu_1 \mathbf{c}^\dagger + f_1,$$

$$\frac{d\mathbf{c}^\dagger}{dt} = g_2' \mathbf{c}^\dagger + \nu_2 \mathbf{c} + f_2. \quad (60)$$

Here the $g_{1,2}'$, $\nu_{1,2}$, and $f_{1,2}$ operators are expressed in terms of Shrödinger operators

$$g_{1,2}' = \sum_{i,j,f} [\pm a_i^\dagger a_f I_{ij}^{\mp} \exp(-i\Delta E_{ij}^{\mp} t/\hbar) I_{jf}^{\pm} R_{jf}^{\pm} \mp a_f^\dagger a_j I_{ij}^{\mp} \times \exp(-i\Delta E_{ij}^{\mp} t/\hbar) I_{fi}^{\pm} R_{fi}^{\pm})],$$

$$\nu_{1,2} = \sum_{i,j,f} [\pm a_i^\dagger a_f I_{ij}^{\mp} \exp(-i\Delta E_{ij}^{\mp} t/\hbar) I_{jf}^{\mp} R_{jf}^{\mp} \mp a_f^\dagger a_j I_{ij}^{\mp} \times \exp(-i\Delta E_{ij}^{\mp} t/\hbar) I_{fi}^{\mp} R_{fi}^{\mp}],$$

$$f_{1,2} = \mp \sum_{i,j} a_i^\dagger a_j I_{ij}^{\mp} \exp(-i\Delta E_{ij}^{\mp} t/\hbar) \quad (61)$$

(here 1 and 2 indices correspond to upper and low signs, respectively). The simple analysis shows that the terms directly proportional to the $\nu_{1,2}$ operators are connected with double photon processes. Here we are not interested in this effect and omit both these terms. Then the system of Eqs. (60) splits in two uncorrelated equations for the $\mathbf{c}$ and $\mathbf{c}^\dagger$ operators. It should also be noted that only nonoscillating (or constant in time) terms play a leading role in operators $g_{1,2}$.

Taking both these remarks into account one obtains

$$\frac{d\mathbf{c}}{dt} = \mathbf{c} g_1 + f_1, \quad (62)$$

$$\frac{d\mathbf{c}^\dagger}{dt} = g_2 \mathbf{c}^\dagger + f_2. \quad (63)$$

Here $g_{1,2}$ operators are defined as

$$g_1 = \sum_{i,j} (D_j^- a_j^\dagger a_j - D_i^+ a_i^\dagger a_i),$$

$$g_2 = \sum_{i,j} (F_j^- a_j^\dagger a_j - F_i^+ a_i^\dagger a_i). \quad (64)$$

$D_j^{\pm}$ and $F_j^{\pm}$ functions are determined by equations

$$D_j^- = (F_j^-)^* = \sum_l \frac{I_{lj}^- I_{jl}^+}{\frac{i}{\hbar}\Delta E_{jl}^- - \eta},$$

$$D_i^+ = (F_i^+)^* = \sum_r \frac{I_{ir}^- I_{ri}^+}{\frac{i}{\hbar}\Delta E_{ri} - \eta}. \quad (65)$$

Note, that Eqs. (62) and (63) are similar to shortened wave equations, in which $f_{1,2}$ are source operators, whereas $g_{1,2}$ are gain operators. Since $n_i = a_i^\dagger a_i$ is the ith level population operator the $g_{1,2}$ operators have inverse population meaning.

To solve Eqs. (62) and (63) we pass from the field operators to c-number functions. Therefore, we multiply Eq. (62) by the $|\Phi_0\rangle$ wave function to the right and Eq. (63) by the $\langle\Phi_0|$ wave function to the left and introduce the notations

$$\mathbf{c}|\Phi_0\rangle = C, \quad g_1|\Phi_0\rangle = G_1'|\Phi_0\rangle, \quad f_1|\Phi_0\rangle = F_1,$$

$$\langle\Phi_0|\mathbf{c}^\dagger = C^*, \quad \langle\Phi_0|g_2 = \langle\Phi_0|G_2', \quad \langle\Phi_0|f_2 = F_2 \quad (66)$$

(note that $|\Phi_0\rangle$ is an eigenfunction of the $n_i = a_i^\dagger a_i$ operator). Thus we obtain two simple equations

$$\frac{dC}{dt} = G_1'C + F_1, \quad \frac{dC^*}{dt} = G_2'C^* + F_2, \quad (67)$$

the solutions of which are

$$C = C_0 \exp(G_1't) + \frac{F_1^0}{G_1' + i\Delta E_{21}^-/\hbar}\left[\exp(G_1't) - \exp\left(-\frac{i}{\hbar}\Delta E_{21}^- t\right)\right],$$

$$C^* = C_0^* \exp(G_2't) + \frac{F_2^0}{G_2' + i\Delta E_{12}^+/\hbar}\left[\exp(G_2 t) - \exp\left(-\frac{i}{\hbar}\Delta E_{12}^+ t\right)\right]. \quad (68)$$

Here C_0 and C_0^* are the integration constants, $F_{1,2}^{(0)} = F_{1,2}(t=0)$, and the $F_{1,2}(t)$ functions are defined by Eqs. (61) and (66). If our device starts from photon vacuum at the time $t=0$ then $C_0 = C_0^* = 0$ and photon expectation ON [Eqs. (44) and (66)] is

$$\overline{N} = \frac{F_1^0 F_2^0}{\left(G_1' + \frac{i}{\hbar}\Delta E_{21}^-\right)\left(G_2' - \frac{i}{\hbar}\Delta E_{21}^-\right)}\left(e^{G_1't} - e^{-(i/\hbar)\Delta E_{21}^- t}\right) \times \left(e^{G_2't} - e^{(i/\hbar)\Delta E_{21}^- t}\right). \quad (69)$$

Now we pass to the real e-beam using the rules of Eqs. (22), (23), and (27). First, we carry out the integration with respect to the E variable in the $G_1' = D_1^- - D_3^+$ and $G_2' = F_1^- - F_3^+$ functions. Using the well-known integration rule.

$$\frac{1}{x \pm i\eta} = P\frac{1}{x} \mp i\pi\delta(x)$$

we remove the singularities in these expressions. Keeping only the real part of the $G_{1,2}'$ functions, one obtains

$$\mathrm{Re}\, G_1' = \mathrm{Re}\, G_2' = g = \frac{1}{2}\frac{c}{n}G. \quad (70)$$

Here the G value is defined by Eq. (36). Then we can perform the procedures of Eqs. (22), (23), and (27) on the $F_1^{(0)}F_2^{(0)}$ product. Finally, we obtain exact expression for photon expectation ON:

$$\overline{N} = 2\pi^2 \rho_0 r_0 \beta_0^2 \sin^2\theta \frac{mc^2}{\varepsilon\hbar\omega}\int dE f(E) \times \frac{|\exp(gt) - \exp[i(\omega - \mathbf{k}\cdot\mathbf{v})t]|^2}{g^2 + (\omega - \mathbf{k}\cdot\mathbf{v})^2} \quad (71)$$

which differs from Eq. (41). Note that one may perform approximative integration with respect to the E variable. It might be relatively easy carried out in the limited cases of large gain

$$G \gg 2k\frac{\Delta}{E_0}\left(\frac{mc}{P_0}\right)^2 \quad (72)$$

and small gain

$$G \ll 2k\frac{\Delta}{E_0}\left(\frac{mc}{P_0}\right)^2. \quad (73)$$

When integrating Eqs. (52) we supposed already that the $\mathbf{c}$, $\mathbf{c}^\dagger$ operators [or C, C^* values (66)] were slowly varying functions of time (such an assumption is equivalent to the small gain approximation), so we must use inequality (73). The computations show that in this case Eq. (71) passes into formula (40). It should be noted, that inequality (73) restricts e-beam density:

$$\rho_0 < \rho_{nl} = 2k\frac{\Delta}{E}\left(\frac{mc}{P_0}\right)^2 \Big/ \frac{dG_1}{d\rho_0}. \quad (74)$$

Here the ρ_{nl} density is obtained from the $G_1 = 2k(\Delta/E_0)(mc/p_0)^2$ equation and the G_1 value is determined by expression (39). It should also be noted that the upper limit of the e-beam density is determined by Fermi's principle. In the general case when the e-beam has both traverse $\Delta_\perp$ and longitudinal $\Delta_\parallel$ momentum spreads one has

$$\rho_0 \leqslant \rho_F \approx \frac{\Delta_\perp^2 \Delta_\parallel}{4\pi^3\hbar^3}. \quad (75)$$

V. SPECTRAL INTENSITY AND ENERGY PROFILES

Now we consider the profiles of spectral intensity and energy produced by low dense (33) and dense (41) e-beams. First of all, we have to fix the observation angle θ and define the medium refractive index $n(\omega)$. As is well-known, $n(\omega)$ is a smooth function of ω frequency far from the special points. Suppose that electrons whose velocity is β_0 produce the photons of ω_0 frequency, so that

$$1 - n(\omega_0)\beta_0\cos\theta = 0.$$

Since the e-beam energy spread is small, in the range close to the ω_0 point one may write to a good accuracy

$$n(\omega) = n_0 + \Delta n = n_0 + (\omega - \omega_0)n_0', \quad (76)$$

where n_0' is the $n(\omega)$ function derivative at the ω_0 point and $\Delta n \ll n_0$. As follows from Eqs. (31) and (32), the velocity and the energy of the electron which produces the ω frequency photon, are, approximately, $v_1 \approx v_c$, $E_1 \approx E_c$, where

$$v_c = v_0\left[1-(\omega-\omega_0)\,\frac{n_0'}{n_0}\right],$$

$$E_c = E_0\left[1-(\omega-\omega_0)\left(\frac{p_0}{mc}\right)^2 \frac{n_0'}{n_0}\right]. \tag{77}$$

After introducing these values into Eq. (33) and taking into account that $E_c \approx E_0$ and $v_c \approx v_0$, one obtains

$$dW = (t-t_0)\,\frac{\sqrt{\pi\,\ln 2}}{\pi^2}\,N^{(e)} e^2 \omega \beta_0^2\,\sin^2\theta\,\frac{n_0}{c}\left(\frac{p_0}{mc}\right)^2 \times \frac{E_0}{\Delta}\,\exp\left[-4\ln 2\,\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right] d\omega do,$$

$$dI = dW/(t-t_0). \tag{78}$$

Thus, the e-beam spontaneous radiated line has Gaussian form with

$$\Delta\omega = \frac{n_0}{n_0'}\,\frac{\Delta}{E_0}\left(\frac{mc}{p_0}\right)^2 \tag{79}$$

width. Note, that the latter value is directly proportional to the e-beam energy spread width Δ and decreases rapidly when the e-beam average energy E_0 increases.

Now we pass to the dense e-beam radiated line consideration. It is convenient to represent the gain expression (36) in two equivalent forms:

$$G = G_{1,2} D_{1,2} R. \tag{80}$$

Here the $G_{1,2}$ values are determined by formulas (39) whereas

$$D_{1,2} = \frac{\lambda}{\lambda_{1,2}}\,\frac{n^2-1}{n}\,\frac{n_{1,2}}{n_{1,2}^2-1} \tag{81}$$

and the multiplicative factor R is

$$R = \sqrt{8e\,\ln 2}\,\frac{\omega-\omega_0}{\Delta\omega}\,\exp\left[-4\ln 2\,\frac{(\omega-\omega_0)^2}{(\Delta\omega)^2}\right]. \tag{82}$$

Since the e-beam energy spread is not large, $\Delta/E_0 \ll 1$, and the Δn value (76) is small, $\Delta n \ll n_0$, $D_1 \approx D_2 \approx 1$ and, as a result, the gain shape is determined by the R function.

It should be noted that (i) $R=0$ at the $\omega=\omega_0$ point, (ii) R achieves its maximum $R_{\max}=1$ and minimum $R_{\min}=-1$ values at the

$$\omega_1 = \omega_0 + \Delta\omega/\sqrt{8\ln 2}, \quad \omega_2 = \omega_0 - \Delta\omega/\sqrt{8\ln 2} \tag{83}$$

points. (iii) In $\omega \gg \omega_0 + \Delta\omega$ and $\omega \ll \omega_0 - \Delta\omega$ portions of the spectrum R function are exponentially small.

Since $\Delta\omega \ll \omega_0$ then, approximately, $\omega_1 \approx \omega_2 \approx \omega_0$, $\lambda_1 \approx \lambda_2 \approx \lambda_0$, and $G_1 \approx G_2 \approx G_0$, where

$$G_0 = 8{,}4\rho_0 r_0 \lambda_0 \beta_0^3 \sin^2\theta\,\frac{n_0^2-1}{n_0}\left(\frac{p_0}{mc}\right)^3\left(\frac{E_0}{\Delta}\right)^2. \tag{84}$$

As was indicated in Sec. III, replacing time t by the $\tau = ct/n$ variable allowed us to pass from the time picture to the space picture. Consider this question in more detail. Suppose that the e-beam has a finite traverse dimension equal to a. Then the length on which the e-beam and photon coupling takes place is $z' = a/\sin\theta$. As a result, the interaction time $t = an/c\,\sin\theta$ and, consequently, the τ parameter is equal to

$$\tau_1 = \frac{ct}{n} = \frac{a}{\sin\theta} = z', \tag{85}$$

i.e., it has interaction length meaning. Taking into account all these remarks, one may rewrite Eqs. (41) as

$$dW = \hbar\omega\,\frac{Q}{G}\,[\exp(G_0 R\tau_1)-1]\,\frac{Vn^3\omega^2 d\omega do}{(2\pi c)^3},$$

$$dI = \frac{\hbar\omega c}{n}\,Q\,\exp(G_0 R\tau_1)\,\frac{Vn^3\omega^2 d\omega do}{(2\pi c)^3}. \tag{86}$$

Let us consider these equations as functions of e-beam density ρ_0. We introduce the definition of the e-beam characteristic density $\rho_{\rm ch}$ on the basis of the equation

$$G_0\tau_1 = 1. \tag{87}$$

Using Eq. (84), one finds

$$\rho_{\rm ch} = \frac{\sin\theta}{a}\left(\frac{dG_0}{dp_0}\right)^{-1} = \left[8{,}4ar_0\lambda_0\,\frac{n_0^2-1}{n_0}\,\beta_0^3\sin^2\theta\left(\frac{p_0}{mc}\right)^3\left(\frac{E_0}{\Delta}\right)^2\right]^{-1}. \tag{88}$$

We shall say that e-beam density ρ_0 is low if

$$\rho_0 < \rho_{\rm ch}. \tag{89}$$

On the contrary, the e-beam is dense if

$$\rho_{nl} > \rho_0 > \rho_{\rm ch}, \tag{90}$$

where the ρ_{nl} density is defined in Eq. (74). And, finally, if $\rho_0 > \rho_{nl}$ then we deal with the high density e-beam. Of course, $\rho_0 \leqslant \rho_F$ [Eq. (75)]. In the first case (89) one may expand the exponent in Eq. (86) in Taylor series. Keeping two first terms only, we find that the radiation produced by a low density e-beam has, mainly, spontaneous nature described by Eq. (78). In the case of the dense e-beam (90), the $G_0\tau_1$ parameter is more than a unit. It is clear, that the $G_0|R|\tau_1$ product is small, $G_0|R|\tau_1 \ll 1$, in the $|\omega-\omega_0| \ll \Delta\omega$ and $|\omega-\omega_0| \gg \Delta\omega$ regions due to the first and third properties of the R function (82). As a result, these portions of the spectrum also have a spontaneous nature described by Eq. (78). The reasons of such phenomena are as follows. As follows from balance equation (35) and Eq. (28) in the range close to the ω_0 point the probability of photon stimulated emission is equal to the probability of photon absorption and they compensate each other. Therefore, this portion of the spectrum has a mainly spontaneous nature.

In the $\omega \gg \omega_0 + \Delta\omega$ and $\omega \ll \omega_0 - \Delta\omega$ regions the radiation is formed by electrons, whose energies are $E_c \ll E_0 - \Delta$ and $E_c \gg E_0 + \Delta$, respectively. For such energies the e-beam spectral density $\rho_0 f\ (E_c)$ is exponentially small and the contribution of stimulated processes is negligible. Note that one may determine the boundaries of the discussed regions more exactly from the $G_0|R|\tau_1 \leq 10^{-1}$ condition. Using the R function definition (82), we find three inequalities

$$\omega_4 \leq \omega \leq \omega_3, \quad \omega \geq \omega_5, \quad \omega \leq \omega_6,$$

where

$$\omega_{3,4} = \omega_0 \pm \Delta\omega/10 G_0 \tau_1 \sqrt{8e\ \ln 2},$$

$$\omega_{5,6} = \omega_0 \pm \Delta\omega\{[\ln(10\sqrt{e} G_0 \tau_1)]/4\ \ln 2\}^{1/2}.$$

The most interesting portions of the spectrum are close to the ω_1 and ω_2 points (83). In the vicinity of ω_1 frequency the R function value is $R(\omega) \approx R(\omega_1) = 1$, so that the parameter $G_0|R|\tau_1 \gg 1$. Therefore, the spectral energy and intensity can be written as

$$dW = \hbar\omega\ \frac{Q}{G}\ \exp(G_0 R \tau_1)\ \frac{V n_0^3 \omega_0^2 d\omega do}{(2\pi c)^3},$$

$$dI = \frac{\hbar\omega c}{n}\ Q\ \exp(G_0 R \tau_1)\ \frac{V n_0^3 \omega_0^2 d\omega do}{(2\pi c)^3}. \qquad (91)$$

It is clear that the spectral shape of these functions differs from Gaussian form. Note also, that both functions achieve their maximums at

$$\omega_1' = \omega_1 - \frac{1}{4}\left(\frac{e}{2\ \ln 2}\right)^{1/2} \frac{\Delta\omega}{G_0 \tau_1}$$

frequency. Since $G_0\tau \gg 1$ and $\Delta\omega \ll \omega_1$, $\omega_1' \approx \omega_1$. Comparison of formulas (91) and (78) shows that the radiation formed by stimulated processes for $[\exp(G_0\tau_1)]/\sqrt{e} G_0 \tau_1$ times exceed the spontaneous radiation. We called this phenomenon the Vavilov-Cherenkov superradiation effect (VChSRE). One can determine the frequency boundaries of the superradiation effect from the $\exp(G_0\tau_1) \geq 10$ condition. Taking into account definition (82), one obtains

$$\omega_7 \leq \omega \leq \omega_8,$$

where

$$\omega_7 = \omega_0 + (\Delta\omega\ \ln 10)/G_1 \tau_1 (8e\ \ln 2)^{1/2},$$

$$\omega_8 = \omega_0 + \Delta\omega\left[\left(\ln \frac{\sqrt{e} G_1 \tau_1}{\ln 10}\right) \Big/ \ \ln 2\right]^{1/2}.$$

Note that in the direct vicinity of the ω_1 point the spectral energy peak (91) has Gaussian form

$$dW = \frac{1}{\sqrt{8\ \ln 2}}\ V\Delta\left(\frac{mc}{\rho_0}\right)^2 \frac{1}{n_0^2 - 1}\left(\frac{n_0}{2\pi c}\right)^3 \omega_0^2 \exp\left[G_0 \tau_1 - 4\ \ln 2\ \frac{(\omega - \omega_1)^2}{(\Delta\omega_1)^2}\right] d\omega do, \qquad (92)$$

the width of which,

$$\Delta\omega_1 = \frac{\Delta\omega}{\sqrt{2 G_0 \tau_1}}, \qquad (93)$$

is appreciably less than the width $\Delta\omega$ of spontaneous radiation (79). The physical meaning of the SR effect may be understood on the basis of balance equation (29) and Eqs. (31) and (32). It is due to the fact that in the vicinity of the ω_1 point [or in the vicinity of E_1' energy (38)] stimulated emission may dominate over spontaneous emission and stimulated absorption.

A very interesting phenomenon occurs in the range close to the ω_2 frequency (83). Here the factor $\omega - \omega_0 < 0$ and one may rewrite the radiation spectral energy dW (86) in a more convenient form:

$$dW = \hbar\omega\ \frac{Q}{|G|}\ [1 - \exp(-G_0|R|\tau_1)]\ \frac{V\omega^2 n^3 d\omega do}{(2\pi c)^3}. \qquad (94)$$

Since the product $G_0\tau_1 \gg 1$ and $|R(\omega)| \approx 1$, then the second term in square brackets is a vanishing value and the saturation of radiated energy takes place:

$$dW = \frac{1}{8\ \ln 2}\ V\Delta\left(\frac{mc}{p_0}\right)^2 \frac{1}{n^2 - 1}\left(\frac{n}{2\pi c}\right)^3 \frac{\Delta\omega}{\omega_0 - \omega}\ d\omega do. \qquad (95)$$

Note that radiation intensity vanishes simultaneously ($dI = dW/dt \to 0$). The physics of the saturation effect follows from balance equation (18). When e-beam radiated energy reaches the level of Eq. (95) then the radiated photon number $dw_{N+1} dt$ is exactly equal to the number of absorbed photons $dw_{N-1} dt$ and the possibility of the equilibrium of the photon state appears. Note that a similar effect takes place when the electromagnetic field interacts with molecular medium and so-called black radiation is built up. We called the radiation described by Eq. (95) Vavilov-Cherenkov black radiation (VChBR). The frequency boundaries of this effect may be determined from the $\exp(-G_0|R|\tau_1) \leq 10^{-1}$ condition. Using definition (82) one obtains

$$\omega_9 \geq \omega \geq \omega_{10},$$

where

$$\omega_9 = \omega_0 - (\Delta\omega\ \ln 10)/G_2 \tau_1 (8e\ \ln 2)^{1/2},$$

$$\omega_{10} = \omega_0 - \Delta\omega\left\{\left[\ln\left(\frac{\sqrt{e} G_2 \tau_1}{\ln 10}\right)\right] \Big/ \ 4\ \ln 2\right\}^{1/2}.$$

It should be noted that the VChBR energy (95) is $\sqrt{e} G_0 \tau_1$ times less than that for spontaneous radiation (78).

VI. PHOTON NUMBER FLUCTUATIONS

One may evaluate photon number fluctuations (as well as the fluctuations of spectral energy and intensity) in the range of VChSRE (92) on the basis of Eqs. (18) and (19). The relative size of the fluctuations is by definition

$$\frac{\Delta N}{\overline{N}}=\frac{(\overline{N^2}-\overline{N}^2)^{1/2}}{\overline{N}}. \tag{96}$$

Note that the first moment $\overline{N}$ was already computed by us and it is determined by Eq. (40), where $G\tau \gg 1$ and the sign of averaging is omitted for simplicity. Let us compute the second moment $\overline{N^2}$. One may rewrite Eq. (19), using this value, as

$$\frac{dN_1^2}{d\tau}=2GN_1^2+4QN_1+Q. \tag{97}$$

Here we introduced new variable $N_1=\sqrt{\overline{N^2}}$ and the G and Q values are defined by Eqs. (36) and (37). A simple integration allows us to pass from differential equation to algebraic one for the N_1 value

$$(2GN_1^2+4QN_1+Q)(GN_1+2Q)\left(2N_1+\frac{1}{2}\right)^{-1} = 4Q^2 \exp(2G\tau). \tag{98}$$

It should be noted that at the time $t=0$, $N_1(t=0)=0$, i.e., our device starts from photon vacuum. The solutions of Eq. (98) may be easily found in the two limited cases where photon expectation number is small, $\overline{N}\ll 1$, and large, $\overline{N}\gg 1$. In the first case one observes the linear growth of the $N_1^2=\overline{N^2}$ value with the respect to the interaction length τ:

$$\overline{N^2}=G\tau,$$

whereas the exponential enhancement of the same value takes place in the second case:

$$\overline{N^2}=4\left(\frac{Q}{G}\right)^2 \exp(2G\tau). \tag{99}$$

Note that the first case corresponds to $\tau \ll 1/G_0$ interaction length; in its turn the second case is valid if $\tau \gg 1/G_0$. Using Eq. (40) one obtains

$$\Delta N=(\overline{N^2}-\overline{N}^2)^{1/2}=\frac{\sqrt{3}Q}{G}\exp(G\tau). \tag{100}$$

The introduction of Eqs. (40) and (100) into Eq. (96) shows that the relative size of radiation fluctuations is more than unit

$$\frac{\Delta N}{\overline{N}}=\sqrt{3}. \tag{101}$$

Thus, the fluctuations level in the superradiation effect (in the framework of the developed approach) is high enough.

VII. CONCLUSION

A theory of dense e-beam radiation was developed. We supposed that the e-beam (which has a Gaussian form energy spread) is uniform in space and our device starts from a photon vacuum. In such a case the radiation field is built up from spontaneous noise, therefore we based our investigation on the QED approach. It is shown that the major parameter which determines the radiation spectral energy and intensity is e-beam density ρ_0. If it is less than some characteristic density ρ_{ch}, then the radiation has a mainly spontaneous nature. One can also obtain the spectral intensity and energy of such radiation directly from the Tamm-Frank formula. In the discussed case they have Gaussian profiles whose width $\Delta\omega$ is directly proportional to the e-beam energy spread width Δ. If the e-beam density is high, $\rho_0>\rho_{ch}$, then a drastic deformation of the spontaneous line profile takes place due to the stimulated processes of photon emission and absorption. We have shown that on the right-hand side of the radiation spectral profile [in the vicinity of the ω_1 frequency (83)] a sharp and narrow peak occurs. Its intensity exceeds appreciably the intensity of spontaneous radiation, whereas its width $\Delta\omega_1$ is much less than $\Delta\omega$—the Vavilov-Cherenkov superradiation effect. A remarkable phenomenon occurs on the left-hand side of the spontaneous radiation line in the vicinity of the ω_2 point (83). When radiation energy achieves the value defined by Eq. (95), then the radiation and absorption processes completely compensate each other and radiation energy saturation takes place (the intensity of such radiation vanishes). We call such a photon state Vavilov-Cherenkov black radiation, since it is similar to the well-known black radiation. Assume that an e-beam with average energy $E_0=12.6$ Mev and relative energy spread $\Delta/E=10^{-3}$ passes gaseous medium with refractive index $n=1.0016$ at $\lambda=1\ \mu$m wavelength. We suppose that the e-beam traverse dimension $a=5$ cm and that we observe radiation at the angle $\theta=3.97\times10^{-2}$ rad. In such a case the electron characteristic density (88) $\rho_{ch}=5\times10^8\ \text{cm}^{-3}$.

If the e-beam is dense [Eq. (90)], $\rho_0=10\rho_{ch}=5\times10^9\ \text{cm}^{-3}$, then the superradiation intensity and energy (92) exceed by three orders the same ones of spontaneous radiation (78) whereas the width $\Delta\omega_1$ (93) is 4.5 less then $\Delta\omega$ [Eq. (79)].

It should be noted that a slight θ-angle tuning leads to a slight tuning of the ω_0 frequency. Comparison of black radiation energy (94) with the same energy of spontaneous radiation (78) (for the parameters chosen above) shows that the latter is 16.5 times more. The obtained results allowed us to write the condition of Cherenkov amplifier operation (43) and to evaluate photon number fluctuations (101). Note also, that the cases of high density electron and photon beams (when nonlinear effects with respect to ρ_0 and N variables are possible) were outside of the present consideration.

ACKNOWLEDGMENTS

The final stage of this work was supported by International Science and Technology Center Grant No. A-87.

[1] I. E. Tamm and I. M. Frank, Dokl. Akad. Nauk SSSR **14**, 107 (1937).

[2] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Nauka, Moscow, 1968).

[3] Juzo Ohkuma, Shuichi Okuda, and Kunihiko Tsumori, Phys. Rev. Lett. **66**, 1967 (1991).

[4] U. Happek, A. J. Sievers, and E. B. Blum, Phys. Rev. Lett. **67**, 2962 (1991).

[5] Young Uk Jeong, Yoshiyuki Kawamura, Koichi Toyoda, Chang Hee Nam, and Sang Soo Lee, Phys. Rev. Lett. **68**, 1140 (1192).

[6] V. M. Harutunian and S. G. Oganesyan, Phys. Rep. **270**, 217 (1996).